Question Layout- Related Rates

QUESTION (Title of the Essay)

ANSWER (Summary of Essay)

SOLUTION (Intro, Body, and Summary of the Essay)

RELATED RATES Essay Writing

$\mathcal{F}_0$ [collect data]

Keywords:

Equations Needed	Given Data	Assemble a one variable function

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time]

$\mathcal{F}_2$ [Summarize]

For Related Rates, we are essentially executing algorithms for physics questions in theory–meaning non-physical—precise.

We look at physics problems differently than math problems and then there is a difference between calculus application and algebraic application of derived calculus formulae found in the standard physics textbooks. What I mean by that is, in your university physics course, you will use derived algebraic equations or formulas vs. assembling functions and using calculus techniques to identify rates of change, minimums, maximums from graphical relation and definition relation. Multiple worlds of physics and math problem solving.

QUESTION [6] 3.9.1

If V is the **volume of a cube** with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

ANSWER [Page A80]

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

SOLUTION

Keywords: volume of a cube

Equations Needed	Solve in One-Variable	One Variable Function
Cuboid (3D Calculus)	For a cuboid, we have	Thus,
$V = LWH = xyz$	$V(x, y, z) = xyz.$	$V(x) = x^3$
Cube $L = W = H = x$	We want this in one variable,	
$V = xxx = x^{1+1+1} = x^3$	$L = x \Rightarrow V = x^3.$	

[FR (Future Reference)] You will be asked the same question for cuboids and other 3D objects using techniques in 3D calculus with partial derivatives and something called Lagrangian Multipliers.

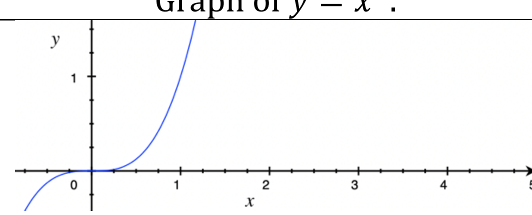
For Related Rates, once you have the single variable function, you simply implicitly differentiate with respect to time. Thus,

$$\frac{d}{dt}[V(x) = x^3] = \frac{dV}{dt} = \frac{d}{dt}x^3 = \frac{d}{dt}[x]^3 = 3[x]^{3-1}\frac{d}{dt}x = 3x^2\frac{dx}{dt}.$$

Graph of $y = x^3$.

ANSWER Therefore, the rate of change is

$$V'(t) = 3x^2 \frac{dx}{dt}.$$



Spherical Ball

QUESTION [1] 3.9.7 (Title of the Essay)

The radius of a **spherical ball** is increasing at a rate of 2 cm/min. At what rate is the **surface area** of the ball increasing when the radius is 8 cm?

ANSWER [Page A80] (Summary of Essay)

$$128\pi\ \frac{\text{cm}^2}{\text{min}}$$

SOLUTION (Intro, Body, and Summary of the Essay)

RELATED RATES Essay Writing $\mathcal{F}_0$ [collect data] Keywords: Sphere, Surface Area

Equations Needed	Values Given	Assemble a one variable function
$V_{\text{sphere}} = \frac{4}{3}\pi r^3$ From the reference page of Edition 7 of the Stewart Series: $SA_{\text{sphere}} = 4\pi r^2$	$\frac{dr}{dt} = 2\ \frac{\text{cm}}{\text{min}}$ $r = 8$ cm $\frac{dSA}{dt} = ?$	$SA(r) = 4\pi r^2$

[RP] The derivative of the volume with respect to the radius, is equal to the surface area. Do you think there is a relation? $\frac{d}{dr}V = \frac{d}{dr}\frac{4}{3}\pi r^3 = \frac{4}{3}(3\pi r^{3-1}) = 4\pi r^2 = SA$.

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time] We are looking for the rate of change of the surface area. Thus, we must differentiate with respect to time, the equation (in oner variable) that represents the surface area.

$$\frac{d}{dt}[SA(r) = 4\pi r^2] \Rightarrow \frac{dSA}{dt} = 4\pi\frac{d}{dt}r^2 = 8\pi r\frac{dr}{dt}$$

$$SA'(t) = 8\pi r\frac{dr}{dt} \Rightarrow \frac{dSA}{dt} = 8\pi[8\text{ cm}]\left[2\ \frac{\text{cm}}{\text{min}}\right] = 8^2(2)\pi\frac{\text{cm}^2}{\text{min}} = 64(2)\pi\frac{\text{cm}^2}{\text{min}} = 128\pi\ \frac{\text{cm}^2}{\text{min}}.$$

$\mathcal{F}_2$ [Summarize]

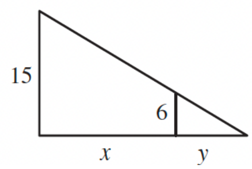
Thus, the rate of change of the surface area is 128π. Or $\frac{dSA}{dt} = 128\pi\ \frac{\text{cm}^2}{\text{min}}$.

QUESTION [1] 3.9.15 (Title of the Essay)

A streetlight is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5ft/s along a straight path. How **<u>fast is the tip of his shadow</u>** moving when he is 40 ft from the pole?

ANSWER [Page A80] (Summary of Essay)

15. (a) The height of the pole (15 ft), the height of the man (6 ft), and the speed of the man (5 ft/s)
(b) The rate at which the tip of the man's shadow is moving when he is 40 ft from the pole
(c)



(d) $\frac{15}{6} = \frac{x+y}{y}$ (e) $\frac{25}{3}$ ft/s

Image from 8th Edition Stewart Calculus [1]

SOLUTION (Intro, Body, and Summary of the Essay)

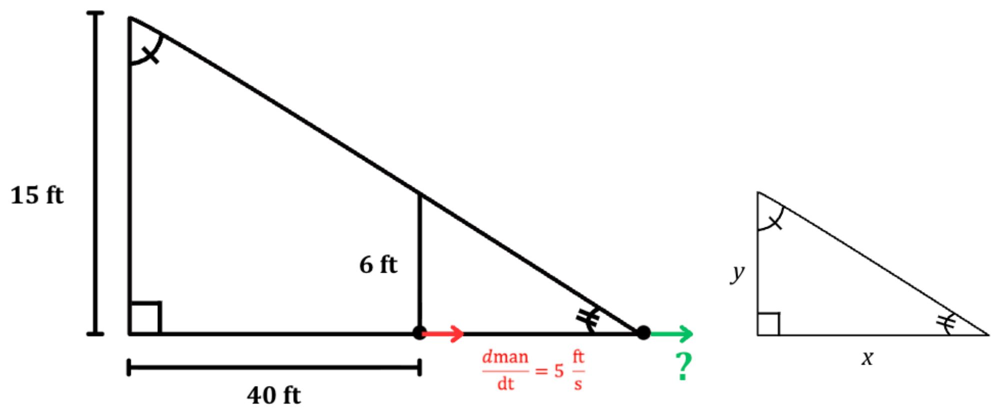
RELATED RATES Essay Writing

$\mathcal{F}_0$ [collect data] Keywords: fast/tip/shadow(triangle)

Equations Needed	Given Data	Assemble a one variable function
We are talking about a triangular relation. Pythagorean $x^2 + y^2 = r^2$ Circumference $C = x + y + r$ Area of Right Triangle $A = \frac{1}{2}bh$	$h_{\text{pole}} = 15\text{ ft}$ $h_{\text{man}} = 6\text{ ft}$ $\frac{d\text{man}}{\text{dt}} = 5\,\frac{\text{ft}}{\text{s}}$ $d_{\text{pole}\to\text{man}} = 40\text{ ft}$	If possible. Else implicit differentiation is necessary.

[EXAM] For related rates, sometimes the algorithmic approach falls into place without thinking; sometimes you need to think. This problem is likely to be an exam question. It is a question that students struggle with because they need a better trigonometric recollection from their studies, but they do not recall the "similar triangle" relation.

What we have here, is a similar triangle relation—that is, the two triangles (right triangles) in question have the same angles but different side/diagonal-lengths.

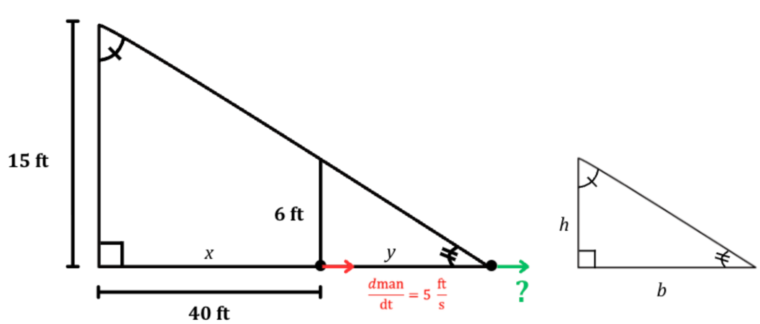


For this question, we are seeking the rate of change of the tip of the shadow. That would be the x coordinate.

For similar triangles, we know that the ratio of the height and base are equal. Thus,

$$\frac{15}{x+40} = \frac{y}{x}$$

Keep in mind that the distance from the pole is changing as well as the distance from the man to the tip of the shadow. So, the previous relation is not going to work because it implies that 40 feet is stationary, which it is not. This is why the author gave extra bits of information in the answer bank. We must account for the distance from the pole to the man as a variable that is changing. We will use the books x and y placement.



For our similar triangle, we have

$$\frac{h_1}{b_1} = \frac{h_2}{b_2} \iff \frac{15}{x+y} = \frac{6}{y}$$

Now, we need to get the equation into a single variable or implicitly differentiate.

$\mathcal{F}_1$ [Implicitly Differentiate w/Respect to Time]

$$\frac{15}{x+y} = \frac{6}{y} \quad \Rightarrow \quad 15y = 6(x+y) \quad \Rightarrow \quad 15y = 6x + 6y \quad \Rightarrow \quad 9y = 6x$$

$$\Rightarrow \quad y = \frac{2}{3}x \quad \Rightarrow \quad \frac{d}{dt}\left[y = \frac{2}{3}x\ \right] \quad \Rightarrow \quad \frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}$$

For the smaller triangle, we see that the hypotenuse and the base are changing but the height is not. I would imagine we need to address this situation. Considering the Pythagorean Theorem, $b^2 + h^2 = r^2$ and implicitly differentiated,

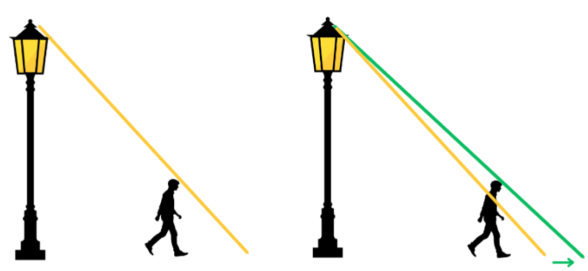
$$2bb' + 2hh' + 2rr' \quad \Rightarrow \quad b\frac{db}{dt} + (0)\frac{dh}{dt} = r\frac{dr}{dt} \Leftrightarrow \quad y\frac{dy}{dt} = r\frac{dr}{dt}$$

From Student X during the live session, they recognized that the rate of change of the shadow is

$$y'(t) = \frac{2}{3}x'(t) = \frac{2}{3}\left(5\,\frac{\text{ft}}{s}\right) = \frac{10}{3}\,\frac{\text{ft}}{\text{s}}.$$

Then, the rate of change of the man (given) $5\,\frac{\text{ft}}{s}$.

Since the shadow is moving at its own rate relative to the height of the man changing the hypotenuse of the smaller triangle, it would make sense that the sum of the rate of the man with the relative rate of the shadow would be the answer. The only question is: what direction is the rate of the shadow moving—left/right?



Take note that the shadow is moving in the positive direction relative to the pole. [RP] If the man was walking towards the pole, would it be subtraction?

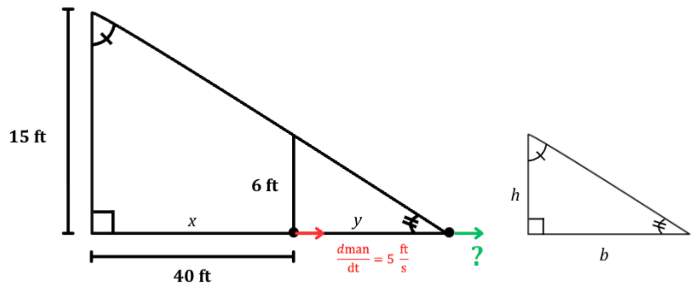
$$\left(\frac{\text{dman}}{\text{dt}} = 5\,\frac{\text{ft}}{\text{s}}\right) + \left(\frac{dy}{dt} = \frac{10}{3}\,\frac{\text{ft}}{\text{s}}\right)$$

$\mathcal{F}_2$ [Summarize]

Thus, the rate of the tip of the shadow is moving at the rate of the man's speed plus the rate of the shadow expanding (let the shadow be stationary so the tip is moving at rate $y'(t)$).

$$\text{rate of the shadow's tip} = 5\,\frac{\text{ft}}{\text{s}} + \frac{10}{3}\,\frac{\text{ft}}{\text{s}} = \frac{(5)(3) + (1)(10)}{(1)(3)}\,\frac{\text{ft}}{\text{s}} = \frac{25}{3}\,\frac{\text{ft}}{\text{s}}.$$

EXAM METHOD (*Practice this until you have memorized and other variations of it by hand.*)



Imagery is not necessary unless requested by professor.

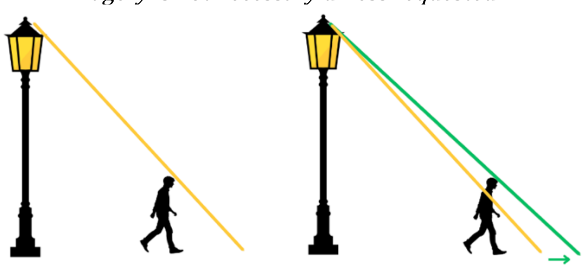
Similar Triangle

$$\frac{h_1}{b_1} = \frac{h_2}{b_2} \iff \frac{15}{x+y} = \frac{6}{y}$$

$$\frac{15}{x+y} = \frac{6}{y} \Rightarrow 15y = 6(x+y) \Rightarrow 15y = 6x + 6y \Rightarrow 9y = 6x$$

$$\Rightarrow y = \frac{2}{3}x \Rightarrow \frac{d}{dt}\left[y = \frac{2}{3}x\right] \Rightarrow \frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}$$

Imagery is not necessary unless requested.



Take note that the shadow is moving in the positive direction relative to the pole. [RP] If the man was walking towards the pole, would it be subtraction?

$$\left(\frac{d\text{man}}{\text{dt}} = 5\ \frac{\text{ft}}{\text{s}}\right) + \left(\frac{dy}{dt} = \frac{10}{3}\ \frac{\text{ft}}{\text{s}}\right)$$

$$\text{rate of the shadow's tip} = 5\ \frac{\text{ft}}{\text{s}} + \frac{10}{3}\ \frac{\text{ft}}{\text{s}} = \frac{(5)(3) + (1)(10)}{(1)(3)}\ \frac{\text{ft}}{\text{s}} = \frac{25}{3}\ \frac{\text{ft}}{\text{s}}.$$

If you are in college, the first part of this solution is practice for professionality and comprehension. The second part (the exam method) is for you to practice with pencil and paper (provided you have a handwritten proctored exam), many times. Always remember, “Doing homework gets you really good at doing homework. Taking exams will help you get better at exams.”—Me.

You should always seek out the other situations in relation to questions done in class or in homework. E.g. (for example), if you have a similar problem with an airplane where the radius is changing. If this question was on the homework, I would anticipate a question with a moving radius/height to be on the exam.